

# CHROMOMAGNETIC INSTABILITY AND THE LOFF STATE IN A TWO FLAVOR COLOR SUPERCONDUCTOR

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## Abstract

We explore the relation between the chromomagnetic instability of a homogeneous two flavor color superconductor and the LOFF state. We perturb the free energy of the 2SC by generating a small net momentum for the quark pair. We find that the imaginary Meissner mass of a particular gluon implies that the LOFF state is energetically favored.

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It is well-known that nuclear matter when is subjected to extreme conditions of temperature and density undergoes phase transitions. New forms of matter have long been sought after, both theoretically and experimentally. Under sufficiently high temperature, nuclear matter is expected to undergo a phase transition to a quark-gluon plasma. At low temperatures but sufficiently high density, however, recent studies suggest a much richer phase structure. One of these phases-color superconductivity- might exist in the core of neutron stars [1, 2]. The baryon density of a color superconductor appropriate to a neutron star is few times higher than the saturation density of nuclear matter. Consequently, perturbative QCD becomes inadequate since the Fermi energy of the quarks is not much larger than  $\Lambda_{\text{QCD}}$ .

An additional complication emerges at the baryon density appropriate for a compact star. The mass of the strange quark  $s$  can no longer be ignored. This causes mismatches between the Fermi sea of the different quark flavors. A large number of exotic Color Superconducting (CSC) phases has been investigated in the literature: these include the CFL ( color flavor locking ) state [3], the LOFF state [4], the 2SC ( two flavor pairing ) and g2SC ( gapless 2SC ) states [5] [6], the gCFL ( gapless CFL ) state [7] and super-normal mixed states [8]. The stability of these phases becomes an important issue to address. The interplay of various homogeneous phases has been investigated recently [9].

Gapless color superconductivity, g2SC or gCFL, is a consequence of breached pairing among the different species of fermions. It was considered initially in the context of an electronic superconductor by Sarma [10], and recently it was applied by Liu and Wilczek in the context of cold atomic gas [11]. If the force responsible for pairing exceeds a critical value, the gap equation for fixed chemical potential for each species admits two branches of solutions. One with fully gaped excitation spectrum and another gapless one. But if we impose the constraint that the total number of each species is fixed, or other constraints such as the charge neutrality [5] etc., the gapless superconductor becomes the only solution. One problem that arises with the gapless solution is the possibility of imaginary Meissner masses [12]. Recently, Huang and Shovkovy found that the squares of the chromomagnetic masses of the five gluons corresponding to the broken generators of the  $SU(3)_c$  group are negative at zero temperature [13]. This result indicates the existence of a chromomagnetic type plasma instability for the underlying color superconducting phase. At zero temperature, this instability extends over the entire branch of the gapless solution and persists in the 2SC phase whenever the mismatch between the Fermi momenta of the different quark flavors is larger than a critical value. Such a chromomagnetic instability has also been discovered in the gCFL phase [14]. The relation of the chromomagnetic instability and the mixed state were explored in Ref. [15].

In this note, we shall examine the relationship between the chromomagnetic instability and the LOFF state. For a 2SC in particular, we find that the chromomagnetic instability associated with the eighth gluon is a sufficient condition for a LOFF state of lower free energy. Complementing the results of [13], we calculate the Meissner screening masses in the region of the phase diagram of QCD, near the second order phase transition to the normal phase. We find that the value of the mismatch at the onset of the chromomagnetic instability agrees with the threshold value of the LOFF state along the transition line, know to an electronic superconductor [16].

We consider the pairing of two quark flavors,  $u$  and  $d$ . The NJL effective Lagrangian

of two flavors coupled to external color and electromagnetic gauge fields in the massless limit is given by

$$\begin{aligned}\mathcal{L} = & -\bar{\psi}\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - igA_\mu^l T_l - ieQA_\mu\right)\psi + \bar{\psi}\gamma_4\mu\psi \\ & + G_S[(\bar{\psi}\psi)^2 + (\bar{\psi}\vec{\tau}\psi)^2] + G_D(\bar{\psi}_C\gamma_5\epsilon^c\tau_2\psi)(\bar{\psi}\gamma_5\epsilon^c\tau_2\psi_C),\end{aligned}\quad (1)$$

where  $\psi$  is the quark field,  $A_\mu^l$  the gluon potential and  $A_\mu$  the electromagnetic potential. The four fermion interaction terms of (23) is taken from Ref. [17], where  $(\epsilon^c)^{mn} = \epsilon^{cmn}$  is a  $3 \times 3$  matrix acting on the red, green and blue color indices. The lagrangian is invariant under  $SU(3)_c \times U(1)_{em}$  transformations. The  $SU(3)_c$  generators and the electromagnetic charge operator are written as

$$T_l = \frac{1}{2}\lambda_l, \quad Q = \frac{1}{6} + \frac{1}{2}\tau_3 \quad (2)$$

where  $\lambda_l$  and  $\vec{\tau}$  are the Gell-Mann and Pauli matrices respectively. The chemical potential reads  $\mu = \bar{\mu} - \delta\tau_3 + \delta'\lambda_8$ , with  $\delta$  and  $\delta'$  being sensitive to the charge and color neutrality respectively. All gamma matrices are hermitian.

The CSC order parameter is set to be

$$\langle \bar{\psi}_f^c \gamma_5 (\psi_C)_{f'}^{c'} \rangle = \Phi_{ff'}^{cc'}, \quad (3)$$

where  $\Phi = -\frac{\Delta}{8G_D}\lambda_2\tau_2$  is a matrix both in color and flavor space, i.e.

$$\Phi_{ff'}^{c'c} = i\frac{\Delta}{8G_D}\epsilon^{c'cb}\epsilon_{ff'} \quad (4)$$

with  $\Delta > 0$ ,  $f = u, d$  and  $c = r, g, b$ . By ignoring the chiral condensate and expanding the NJL Lagrangian to linear order in the fluctuation

$$\bar{\psi}_f^c \gamma_5 (\psi_C)_{f'}^{c'} - \Phi_{ff'}^{cc'}, \quad (5)$$

we derive the following mean field expression for the NJL lagrangian

$$\mathcal{L}_{MF} = -\frac{\Delta^2}{4G_D} - \bar{\psi}\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - igA_\mu^l T^l - ieQA_\mu\right)\psi + \bar{\psi}\gamma_4\mu\psi + \Delta[-\bar{\psi}_C\gamma_5\lambda_2\tau_2\psi + \bar{\psi}\gamma_5\lambda_2\tau_2\psi_C]. \quad (6)$$

The thermodynamic potential  $\Omega$  is given by the Euclidean path integral

$$\Omega = k_B T \ln \left[ \int [d\psi d\bar{\psi}] \exp \left( \int d^4x \mathcal{L}_{MF} \right) \right] \quad (7)$$

with  $0 < x_4 < (k_B T)^{-1}$  and can be expanded in terms of one loop diagrams. The corresponding free energy,  $\mathcal{F} \equiv -\Omega$  will be minimized at equilibrium.

The Nambu-Gorkov form of the inverse quark propagator in the super phase takes the form

$$\mathcal{S}^{-1}(P) = \begin{pmatrix} \not{P} + \mu\gamma_4 & \Delta\gamma_5\lambda_2\tau_2 \\ -\Delta\gamma_5\lambda_2\tau_2 & \not{P} - \mu\gamma_4 \end{pmatrix} \quad (8)$$

where  $\not{P} = -i\gamma_\mu P_\mu$ ,  $P = (\vec{p}, -\nu)$  and  $\nu = (2n+1)\pi k_B T$  is the Matsubara frequency.  $\psi_C = C\tilde{\bar{\psi}}$  is the charge conjugate of  $\psi$  with  $C = i\gamma_2\gamma_4$ . Each block of the matrix quark propagator (8) is itself a matrix in color-flavor space. The NG representation of the quark-gluon and quark-photon vertex is

$$\Gamma_\mu^l = \begin{pmatrix} \gamma_\mu \mathcal{T}_l & 0 \\ 0 & -\gamma_\mu \tilde{\mathcal{T}}_l \end{pmatrix}, \quad (9)$$

with  $l = 1, 2, \dots, 9$  and the tilde standing for transpose. We have

$$\mathcal{T}_l = T_l \quad (10)$$

for  $l = 1, \dots, 7$ ,

$$\mathcal{T}_8 = T_8 \cos \theta + \frac{e}{g} Q \sin \theta \quad (11)$$

and

$$\mathcal{T}_9 = -\frac{g}{e} T_8 \sin \theta + Q \cos \theta \quad (12)$$

with  $\cos \theta = \frac{\sqrt{3}g}{\sqrt{3g^2+e^2}}$  and  $\sin \theta = \frac{e}{\sqrt{3g^2+e^2}}$ . Here we have introduced the gluon-photon mixing [18]. The condensate is invariant under the residual gauge transformations  $SU(2) \times \mathcal{U}(1)$ , where  $SU(2)$  is generated by  $T_1, T_2, T_3$  and  $\mathcal{U}(1)$  by  $\mathcal{T}_9$ .

A particularly convenient representation of the quark propagator and the vertex can be obtained by performing a unitary transformation generated by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & \tau_2 \end{pmatrix}. \quad (13)$$

We have

$$\begin{aligned} \mathcal{S}'^{-1}(P) &\equiv U \mathcal{S}^{-1}(P) U^\dagger \\ &= \begin{pmatrix} \not{P} + \bar{\mu}\gamma_4 - \delta\gamma_4\tau_3 + \delta'\lambda_8 & \Delta\gamma_5\lambda_2 \\ -\Delta\gamma_5\lambda_2 & \not{P} - \bar{\mu}\gamma_4 - \delta\gamma_4\tau_3 - \delta'\lambda_8 \end{pmatrix} \end{aligned} \quad (14)$$

and

$$\Gamma_\mu^{\prime l} = \begin{pmatrix} \gamma_\mu T_l & 0 \\ 0 & -\gamma_\mu \tau_2 \tilde{T}_l \tau_2 \end{pmatrix}. \quad (15)$$

In the remaining of this paper we shall use this representation and we shall suppress the primes from  $\mathcal{S}$  and  $\Gamma$ . We notice that  $\mathcal{S}(P)$  in the new representation commutes with  $\tau_3$ .

The squared Meissner mass is defined by

$$(m^2)^{\prime l} = \frac{1}{2} \lim_{\vec{K} \rightarrow 0} (\delta_{ij} - \hat{k}_i \hat{k}_j) \Pi_{ij}^{\prime l}(K) = \frac{1}{2} (\delta_{ij} - \hat{z}_i \hat{z}_j) \Pi_{ij}^{\prime l}(0) \quad (16)$$

where

$$\Pi_{ij}^{\prime l}(K) = -\frac{k_B T}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \text{Tr} \Gamma_i^{\prime l} \mathcal{S}(P) \Gamma_j^l \mathcal{S}(P+K) \quad (17)$$

is the gluon polarisation tensor with  $K = (\vec{k}, -\omega)$ . Furthermore we have  $(m^2)^{ll'} = 0$  for  $\Delta = 0$ .

A simplification follows from the observation that the isospin part of the charge operator (2) contributes to the Meissner masses a total derivative, i.e.

$$-\frac{k_B T}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \text{Tr} \gamma_j \tau_3 \mathcal{S}(P) M \mathcal{S}(P) = -i \frac{k_B T}{2} \int \frac{d^3 \vec{p}}{(2\pi)^3} \frac{\partial}{\partial p_j} \text{Tr} \tau_3 M \mathcal{S}(P) \quad (18)$$

where  $M$  is an arbitrary matrix. Its contribution then can be omitted and the charge operator may be effectively replaced by  $Q = \frac{1}{6}$ .

It follows from the residual  $SU(2)$  symmetry that  $(m^2)^{ll'}$  is diagonal with respect to  $l'$  and  $l$  that label the standard Gell-Mann matrices. We proceed to divide the set of the nine generators into three subsets. Subset I includes the unbroken generators  $\mathcal{T}_1$ ,  $\mathcal{T}_2$  and  $\mathcal{T}_3$ , subset II includes the broken generators  $\mathcal{T}_4$ ,  $\mathcal{T}_5$ ,  $\mathcal{T}_6$  and  $\mathcal{T}_7$  while subset III includes the broken  $\mathcal{T}_8$  and the unbroken  $\mathcal{T}_9$  generators. Under an  $SU(2)$  transformation,  $\mathcal{T}_l \rightarrow u \mathcal{T}_l u^\dagger$ , the generators in the subset I transform as the adjoint representation, the generators in the subset II as the fundamental one while the generators in the subset III remain invariant. Therefore the indices  $l'$  and  $l$  which correspond to generators from different subsets do not mix in the expression for  $(m^2)^{ll'}$ . The mass matrix within subset I vanishes following the standard arguments of gauge invariance. As to the subset II, we proceed to relabel its members according to  $\mathcal{J}_1 = \mathcal{T}_4$ ,  $\mathcal{J}_1 = \mathcal{T}_5$ ,  $\mathcal{J}_2 = \mathcal{T}_6$  and  $\mathcal{J}_2 = \mathcal{T}_7$ . Introducing

$$\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (19)$$

we have

$$\mathcal{J}_\alpha = \begin{pmatrix} 0 & \phi_\alpha \\ \phi_\alpha^\dagger & 0 \end{pmatrix}, \quad (20)$$

and similar expressions for  $\mathcal{J}_{\bar{\alpha}}$  with  $\bar{\phi}_\alpha = -i\phi_\alpha$ , where the block decomposition is with respect to color indices. Since  $\mathcal{J}_\alpha$  is symmetric and  $\mathcal{J}_{\bar{\alpha}}$  is antisymmetric, there is no mixing between  $\mathcal{J}_\alpha$  and  $\mathcal{J}_\beta$ , following from the properties of the matrix  $C = i\gamma_2\gamma_4$ , i.e.  $C\tilde{\gamma}_\mu C^{-1} = -\gamma_\mu$  and  $C\tilde{\mathcal{S}}(P)C^{-1} = -\mathcal{S}(P)$ . The matrix element  $(m^2)^{\alpha\beta}$  or  $(m^2)^{\bar{\alpha}\bar{\beta}}$  can be written as a sum of terms  $(\phi^\alpha)^\dagger M \phi^\beta$  or  $(\phi^{\bar{\alpha}})^\dagger M \phi^{\bar{\beta}}$  with the same  $SU(2)$  invariant  $M$ . Therefore we have  $(m^2)^{\alpha\beta} = (m^2)^{\bar{\alpha}\bar{\beta}} \propto \delta_{\alpha\beta}$ . Finally in subset III, the color matrices never mix the red and green components with the blue ones. The latter do not carry a condensate. It follows then that we need only to work within the red-green subspace, in which  $\mathcal{T}^9$  does not contribute and  $\mathcal{T}^8$  may be replaced by a factor  $\frac{1}{6}\sqrt{3g^2 + e^2}$ .

Next we proceed to define the susceptibility as the response of the free energy to a small net momentum of the Cooper pair. This amounts in replacing the order parameter (5) with

$$\langle \bar{\chi} \gamma_5 \chi_C \rangle = \Phi \quad (21)$$

where

$$\chi(\vec{r}) = e^{i\vec{q}\cdot\vec{r}} \psi(\vec{r}) \quad (22)$$

and to form the NG basis with respect to the new field  $\chi$  and its charge conjugate  $\chi_C = C\tilde{\chi}$ . The net momentum carried by the Cooper pair is then  $2\vec{q}$ . The NJL lagrangian

(6) expressed in terms of  $\chi$  reads

$$\begin{aligned}\mathcal{L} &= -\bar{\chi}\gamma_\mu\left(\frac{\partial}{\partial x_\mu} - igA_\mu^l T^l - ieQA_\mu\right)\chi - i\bar{\chi}\vec{\gamma}\cdot\vec{q}\chi + \bar{\chi}\gamma_4\mu\chi \\ &+ G_S[(\bar{\chi}\chi)^2 + (\bar{\chi}\vec{\tau}\chi)^2] + G_D(\bar{\chi}_C\gamma_5\epsilon^c\tau_2\chi)(\bar{\chi}\gamma_5\epsilon^c\tau_2\chi_C),\end{aligned}\quad (23)$$

and gives rise to a new vertex  $\vec{q}\cdot\vec{\Gamma}$  with

$$\vec{\Gamma} = \begin{pmatrix} \vec{\gamma} & 0 \\ 0 & -\vec{\gamma} \end{pmatrix} \quad (24)$$

in NG basis. The free energy can be expanded according to increasing powers of  $q$ ,

$$\mathcal{F} = \mathcal{F}_0 + \frac{1}{2}\kappa q^2 + \dots \quad (25)$$

We call the coefficient  $\kappa$  momentum susceptibility. A signal that the LOFF state has lower free energy than the 2SC state is a negative momentum susceptibility. We have

$$\kappa = -\frac{1}{3}\delta_{ij}\frac{k_B T}{2}\int\frac{d^3\vec{p}}{(2\pi)^3}\text{Tr}\Gamma_i\mathcal{S}(P)\Gamma_j\mathcal{S}(P). \quad (26)$$

We observe that the relationship

$$m_8^2 = \frac{1}{12}\left(g^2 + \frac{e^2}{3}\right)\kappa \quad (27)$$

is valid for all temperatures in the super phase. Therefore the chromomagnetic instability of the gluon that corresponds to the 8th color generator is equivalent to the instability against LOFF pairing.

In order to compare our results with the existing literature on LOFF pairing [16], we calculate explicitly the Meissner masses and the momentum susceptibility near the second order transition temperature where  $\Delta \ll k_B T$ . We shall make the approximation that  $\delta' = 0$ , consistent with the numerical study in Ref. [5]. As a result  $\bar{\mu} = \frac{1}{2}(\mu_u + \mu_d)$  and  $\delta = \frac{1}{2}(\mu_d - \mu_u)$ . While it is straightforward to obtain a closed form of the quark propagator for an arbitrary gap, it will suffice to expand it to second order in  $\Delta$ , with  $\Delta$  being the solution of the gap equation as  $T \rightarrow T_c$ . After some algebra, we find that

$$(m^2)^{l'l} = \frac{1}{2}g^2\Delta^2[C_1^{l'l}I + C_2^{l'l}J]. \quad (28)$$

The quantities  $I$  and  $J$  are given by

$$\begin{aligned}I &= -\frac{1}{2}k_B T \sum_\nu \int \frac{d^3\vec{p}}{(2\pi)^3} (\delta_{ij} - \hat{k}_i\hat{k}_j) \text{tr} \left[ \gamma_i S_+^u(P) \gamma_j S_+^u(P) S_-^d(P) S_+^u(P) \right. \\ &\quad \left. + \gamma_i S_+^d(P) \gamma_j S_+^d(P) S_-^u(P) S_+^d(P) \right].\end{aligned} \quad (29)$$

and

$$\begin{aligned}J &= \frac{1}{4}k_B T \sum_\nu \int \frac{d^3\vec{p}}{(2\pi)^3} (\delta_{ij} - \hat{k}_i\hat{k}_j) \text{tr} \left[ \gamma_i S_+^u(P) S_-^d(P) \gamma_j S_-^d(P) S_+^u(P) \right. \\ &\quad \left. + \gamma_i S_+^d(P) S_-^u(P) \gamma_j S_-^u(P) S_+^d(P) \right],\end{aligned} \quad (30)$$

with

$$S_{\pm}^f(P) = \frac{1}{\not{P} \pm \mu_f \gamma_4} \quad (31)$$

and  $S_{\mp}^f(P) = -S_{\pm}^f(-P)$ . Using the relation

$$\frac{\partial}{\partial p_j} S_{\pm}^f(P) = i S_{\pm}^f(P) \gamma_j S_{\pm}^f(P) \quad (32)$$

and integrating by parts, we can show that  $I = J$ .

The group theoretic factors

$$C_1'^l = 4\text{tr}(\mathcal{T}_{l'} \mathcal{T}_l + \mathcal{T}_l \mathcal{T}_{l'}) \lambda_2^2 \quad (33)$$

and

$$C_2''^l = 4\text{tr} \mathcal{T}_{l'} \lambda_2 \tilde{\mathcal{T}}_l \lambda_2 \quad (34)$$

in the standard representation of the Gell-Mann matrices are given by

$$C_1 = \text{diag.} \left( 2, 2, 2, 1, 1, 1, 1, \frac{2}{3} \left( 1 + \frac{e^2}{3g^2} \right), 0 \right) \quad (35)$$

and

$$C_2 = \text{diag.} \left( -2, -2, -2, 0, 0, 0, 0, \frac{2}{3} \left( 1 + \frac{e^2}{3g^2} \right), 0 \right). \quad (36)$$

Combining Eq. (28), (35) and (36), we obtain the following expressions for the Meissner masses of the gluons

$$m^2|_{4,5,6,7} = \frac{1}{2} g^2 \Delta^2 J, \quad (37)$$

$$m^2|_8 = \frac{2}{3} \left( g^2 + \frac{e^2}{3g^2} \right) \Delta^2 J. \quad (38)$$

corresponding to the broken generators  $\mathcal{T}_4$ ,  $\mathcal{T}_5$ ,  $\mathcal{T}_6$ ,  $\mathcal{T}_7$  and  $\mathcal{T}_8$  of the  $SU(3)_c \times U(1)_{\text{em}}$  respectively.

Carrying out the momentum integral in Eq.(30), we obtain the expression

$$\begin{aligned} J &= \frac{\mu^2 k_B T}{3\pi} \text{Re} \left[ \sum_{\nu > 0} \frac{1}{(\nu + i\delta)^3} + \frac{3}{\mu^2} \sum_{0 < \nu < \Lambda} \frac{1}{\nu + i\delta} \right] \\ &= -\frac{\mu^2}{24\pi^4 k_B^2 T^2} \text{Re} \psi'' \left( \frac{1}{2} + i \frac{\delta}{2\pi k_B T} \right) \\ &\quad - \frac{1}{2\pi^2} \left[ \text{Re} \psi \left( \frac{1}{2} + i \frac{\delta}{2\pi k_B T} \right) + \gamma_E - \sum_{n=1}^N \frac{1}{n} \right], \end{aligned} \quad (39)$$

where  $\psi(z) = \frac{d \ln \Gamma(z)}{dz}$ ,  $\gamma_E = 0.5772\dots$  and we have introduced an UV cutoff  $\Lambda$  in the summation over the Matsubara energies with  $\Lambda = 2\pi k_B T \left( N + \frac{1}{2} \right)$ .  $\Lambda$  represents the energy scale of the Cooper pair. Until this point we have made no assumptions about the size of the mismatch  $\delta$  when compared to the strength of the average chemical potential

$\bar{\mu}$ . The typical value of  $\delta$  that results from the charge neutrality condition is not much smaller than  $\bar{\mu}$  [5].

For  $\delta \ll \mu$ ,  $\Lambda \ll \mu$  and  $k_B T \ll \mu$ , we have

$$J \simeq -\frac{\mu^2}{48\pi^4 k_B^2 T^2} \text{Re}\psi''\left(\frac{1}{2} + i\frac{\delta}{2\pi k_B T}\right) \quad (40)$$

and the term dropped is smaller by an order of  $O\left(\left(\frac{k_B T}{\mu}\right)^2 \ln \frac{\mu}{\Lambda}\right)$ . The Meissner masses of the gluons corresponding to the broken generators  $\mathcal{T}_4$ ,  $\mathcal{T}_5$ ,  $\mathcal{T}_6$  and  $\mathcal{T}_7$  are given by

$$m^2|_{4,5,6,7} = -\frac{g^2 \mu^2 \Delta^2}{96\pi^4 (k_B T)^2} \text{Re}\psi''\left(\frac{1}{2} + i\frac{\delta}{2\pi k_B T}\right), \quad (41)$$

while the Meissner mass for the gluon corresponding to the broken generator  $\mathcal{T}_8$  by

$$m^2|_8 = -\frac{g^2 \mu^2 \Delta^2}{72\pi^4 (k_B T)^2} \left(1 + \frac{e^2}{3g^2}\right) \text{Re}\psi''\left(\frac{1}{2} + i\frac{\delta}{2\pi k_B T}\right). \quad (42)$$

The momentum susceptibility under the same approximation reads

$$\kappa = -\frac{\mu^2 \Delta^2}{6\pi^4 (k_B T)^2} \text{Re}\psi''\left(\frac{1}{2} + i\frac{\delta}{2\pi k_B T}\right). \quad (43)$$

In the absence of the mismatch,  $\delta = 0$ , we recover, with the aid of the formula  $\psi''\left(\frac{1}{2}\right) = -14\zeta(3)$ , the familiar parameters of the Ginzburg-Landau free energy [19],[20],

$$m^2|_{4,5,6,7} = \frac{7\zeta(3)g^2 \mu^2 \Delta^2}{48\pi^4 (k_B T)^2}, \quad (44)$$

$$m^2|_8 = \frac{7\zeta(3)g^2 \mu^2 \Delta^2}{36\pi^4 (k_B T)^2} \left(1 + \frac{e^2}{3g^2}\right) \quad (45)$$

and

$$\kappa = \frac{7\zeta(3)\mu^2 \Delta^2}{3\pi^4 (k_B T)^2}. \quad (46)$$

The real part of the function  $-\psi''\left(\frac{1}{2} + i\frac{\delta}{2\pi k_B T}\right)$  is positive for small mismatch and changes its sign at  $\frac{\delta}{2\pi k_B T} \simeq 0.3041$ . This is the critical value which signals the emergence of the chromomagnetic instability. Concurrently, the momentum susceptibility becomes negative signaling the LOFF instability.

The pairing temperature for a mismatched Fermi sea of electrons at given  $\delta$  and  $q$  has been worked out and is given by the solution to the transcendental equation [16]

$$\ln \frac{T_{\delta,q}}{T_0} = \frac{\pi k_B T_{\delta,q}}{q} \text{Im} \ln \frac{\Gamma\left(\frac{1}{2} - i\frac{\delta+q}{2\pi k_B T_{\delta,q}}\right)}{\Gamma\left(\frac{1}{2} - i\frac{\delta-q}{2\pi k_B T_{\delta,q}}\right)} - \gamma_E - 2 \ln 2. \quad (47)$$



The transition temperature for a given  $\delta$  is the maximum of  $T_{\delta,q}$  with respect to  $q$ . The small  $q$  expansion of  $T_{\delta,q}$  reads

$$\begin{aligned} \ln \frac{T_{\delta,q}}{T_c} &= -\gamma_E - 2 \ln 2 - \operatorname{Re} \psi \left( \frac{1}{2} + i \frac{\delta}{2\pi k_B T_{\delta,0}} \right) \\ &+ \frac{q^2}{24\pi^2 k_B^2 T_{\delta,0}^2} \operatorname{Re} \psi'' \left( \frac{1}{2} + i \frac{\delta}{2\pi k_B T_{\delta,0}} \right) + \dots \end{aligned} \quad (48)$$

The LOFF pairing prevails when the coefficient of  $q^2$  becomes positive, which is exactly the point where the expression for the Meissner masses changes sign. Upon substitution of the value  $\frac{\delta}{2\pi k_B T_{\delta,0}} \simeq 0.3041$ , we obtain that

$$\frac{\delta}{\Delta_0} = 0.6082, \quad (49)$$

[16] which is the threshold value of the LOFF window along the transition line, where  $\Delta_0$  is the gap at  $T = 0$  in the absence of the mismatch and  $\frac{\Delta_0}{k_B T_0} = \pi e^{-\gamma_E}$ , following the standard BCS relation.

What we have considered so far is the response of the free energy of a 2SC or a g2SC to a virtual displacement of a small net momentum  $\vec{q}$  in the direction of the simple LOFF pairing. The condition of the corresponding instability is the same as the chromomagnetic instability associated with the rotated 8-th gluon. The true minimum of the free energy with the LOFF pairing requires that

$$\vec{\nabla}_{\vec{q}} \mathcal{F} = 0 \quad (50)$$

and that  $\frac{\partial^2 \mathcal{F}}{\partial q_i \partial q_j}$  is non negative. The existence of the solution has been demonstrated in [16] for an electronic system ( though the gradient function (50) at  $T = 0$  may be discontinuous at the value of  $q$  when the branch of the gapped excitations at  $q = 0$  become gapless). It was found that the value of  $q$  of the solution away from the transition temperature becomes quickly comparable to the inverse coherence length at zero temperature. An important property of the minimum is that the electric current vanishes there and consequently there is no induced magnetic field. For the color superconductor considered here, the condition (50) implies the vanishing tadpole diagram, i.e.

$$\langle \bar{\psi} \vec{\gamma} \psi \rangle = 0. \quad (51)$$

What about color and electric currents? In order to answer this question, we write down the color-electric current in terms of the exact quark propagator at finite  $\vec{q}$  and  $\delta$ , i.e.

$$\vec{J}_l = -k_B T \sum_{\nu} \int \frac{d^3 \vec{p}}{2\pi^3} \operatorname{Tr} \vec{\Gamma}^l \mathcal{S}_{\vec{q}}(P) \quad (52)$$

where

$$\mathcal{S}_{\vec{q}}^{-1}(P) = \begin{pmatrix} \not{P} - i\vec{\gamma} \cdot \vec{q} + \bar{\mu}\gamma_4 - \delta\gamma_4\tau_3 + \delta'\lambda_8 & \Delta\gamma_5\lambda_2 \\ -\Delta\gamma_5\lambda_2 & \not{P} + i\vec{\gamma} \cdot \vec{q} - \bar{\mu}\gamma_4 - \delta\gamma_4\tau_3 - \delta'\lambda_8 \end{pmatrix}. \quad (53)$$

Again, the isospin part of the charge operator contributes only a total derivative because of the relation

$$Tr\gamma_j\tau_3\mathcal{S}_{\vec{q}}(P) == -i\frac{\partial}{\partial p_j} \ln \det\mathcal{S}_{\vec{q}}(P)|_{\tau_3=1} + i\frac{\partial}{\partial p_j} \ln \det\mathcal{S}_{\vec{q}}(P)|_{\tau_3=-1} \quad (54)$$

and may be omitted. The residual  $SU(2) \times \mathcal{U}(1)$  symmetry implies that  $\vec{J}_l = 0$  ( $l \neq 8$ ) for all  $q$  and

$$\vec{J}_8 = \frac{1}{6}\sqrt{3g^2 + e^2} < \bar{\psi}\vec{\gamma}\psi > = 0. \quad (55)$$

for the value of  $q$  that satisfies eq. (50). Therefore all currents vanish at the LOFF minimum. While the chromomagnetic instability associated with the 8th gluon has been removed, we need to check if the chromomagnetic instability associated with the other color components vanishes as well.

In principle, there exist several directions that lead to a LOFF state corresponding to imaginary Meissner masses for different gluons, along which a virtual displacement lowers the free energy of the condensate. This amounts in replacing the transformation (22) by

$$\chi(\vec{r}) = e^{iT\vec{q}\cdot\vec{r}}\psi(\vec{r}) \quad (56)$$

where  $T$  is the combination of the group generators that corresponds to the negative eigenvalue of the mass square matrix  $(m^2)^{ll'}$ . This is true both for the other gluons ( $l = 4, 5, 6, 7$ ) of a 2SC superconductor and for the chromomagnetic instability of a gCFL superconductor. It is not clear, however, if the condition (50) guarantees the vanishing of all the components of the color current. If not, the true minimum will be considerably more complicated than the LOFF state since a nonzero expectation value of the gluon field will be induced.

Before concluding this paper, we would like to argue that the charge neutrality condition may be implemented in the LOFF state as long as it can be implemented in the g2SC superconductor. The gap equation and the charge neutrality condition can be written as

$$\frac{\partial\mathcal{F}}{\partial\Delta} = \frac{\partial\mathcal{F}}{\partial\delta} = 0 \quad (57)$$

If we treat the second term of (25) perturbatively, the deformation of the solution  $(\Delta, \delta)$  to (57) from the one at  $q = 0$  is of order  $O(q^2)$  provided that the determinant

$$\left( \begin{array}{cc} \frac{\partial^2\mathcal{F}_0}{\partial\Delta^2} & \frac{\partial^2\mathcal{F}_0}{\partial\Delta\partial\delta} \\ \frac{\partial^2\mathcal{F}_0}{\partial\Delta\partial\delta} & \frac{\partial^2\mathcal{F}_0}{\partial\delta^2} \end{array} \right) \neq 0 \quad (58)$$

at  $q = 0$ . The numerical solution of [5] indicates that the stationary point defined by (57) at  $q = 0$  and  $T = 0$  is a saddle point with respect to  $\Delta$  and  $\delta$ , in which case the condition (58) is true. Therefore the impact of the charge neutrality condition on the free energy for a small nonzero  $q$  is of order  $O(q^4)$  and as a result cannot compete with the  $q^2$  term of (25). As  $q$  gradually increases from zero, eqs.(57) defines a charge neutral trajectory in the three dimensional parameter space of  $\Delta$ ,  $\delta$  and  $q$  that intersects with the plane of  $q = 0$  at the solution of Ref. [5]. Since the free energy is a continuous function

and is bounded from below, there exists a minimum at  $q \neq 0$  along the trajectory if  $\kappa$  or equivalently  $m_g^2$  is negative.

In this letter, we have explored the relation between the newly discovered chromomagnetic instability of a homogeneous color superconductor and the tendency towards a LOFF pairing by providing a net momentum to a Cooper pair to the free energy landscape. Our procedure is similar to that of Wu and Yip for a nonrelativistic superfluid [12]. We showed that whenever the square of the Meissner mass of the 8th gluon becomes negative there is a LOFF state with lower free energy. Further investigation is still required in order to decide whether this LOFF state represents a true equilibrium phase of the quark matter at moderate baryon density. Future work will concentrate on calculating the Meissner masses at the LOFF minimum, examining the effect of charge neutrality, and comparing the free energy of the LOFF minimum to the free energy of other exotic phases proposed in the literature. We hope to report on these issues in the near future.

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